THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW8 Solution

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1. (P.246 Q5)

   Case 2: $x = 0$: Then
$$\lim_{n\to\infty} \frac{\sin nx}{1+nx} = \frac{0}{1+0} = 0$$

   Case 3: $0 < x < +\infty$: Since $|\sin nx| \leq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \frac{1}{1+nx} = 0$, we have
$$\lim_{n\to\infty} \frac{\sin nx}{1+nx} = 0$$

2. (P.247 Q15)

   (i) Fix $a > 0$, then by Q5, for all $x \in [a, +\infty)$, $\lim_{n\to\infty} \frac{\sin nx}{1+nx} = 0$. We claim the convergence is uniform in $[a, +\infty)$:

   Given $\epsilon > 0$, since $\lim_{n\to\infty} \frac{1}{1+na} = 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{1+Na} < \epsilon$. Then for all $n \geq N$, $x \in [a, +\infty)$,
$$\begin{aligned} \left|\frac{\sin nx}{1+nx}\right| &\leq \frac{1}{1+Na} \\ &< \epsilon \end{aligned}$$

   Therefore, the convergence is uniform in $[a, +\infty)$.

   (ii) We claim that the convergence is not uniform in $[0, +\infty)$: By Q5, if the convergence were uniform, the uniform limit function would be given by $f(x) = 0$ for all $x \in [0, +\infty)$.

   We use Lemma 8.15 of the textbook to show that $f_n(x) = \frac{\sin nx}{1+nx}$ does not converge to f:
   Choose $\epsilon_0 = \frac{1}{1+\pi}$, $n_k = k$, $x_k = \frac{\pi}{2k}$. Then
$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= \left|\frac{\sin \frac{\pi}{2}}{1+\frac{\pi}{2}}\right| \\ &= \frac{1}{1+\frac{\pi}{2}} \\ &> \frac{1}{1+\pi} = \epsilon_0 \end{aligned}$$

Therefore, the convergence is not uniform.

3. (P.247 Q22)

   To show the uniform convergence of f_n to f, note that $f_n(x) - f(x) = (x + \frac{1}{n}) - x = \frac{1}{n}$, and hence $\|f_n - f\|_{\mathbb{R}} = \frac{1}{n} \to 0$ as $n \to \infty$. Therefore, by Lemma 8.1.8 of the textbook, f_n converges uniformly to f on $\mathbb{R}$.

   To show f_n^2 does not converge uniformly on $\mathbb{R}$, by Lemma 8.1.10 of the textbook, it suffices to find some $\epsilon_0 > 0$ such that for all $N \in \mathbb{N}$, there exists $m, n \geq N$ and $x \in \mathbb{R}$ such that

   $$\left|f_n^2(x) - f_m^2(x)\right| \geq \epsilon_0$$

   Let $\epsilon_0 = 1$, for all $N \in \mathbb{N}$, chooses $m = 2N, n = N, x = N$, then

   $$\begin{aligned}\left|f_n^2(x) - f_m^2(x)\right| &= \left|(x + \frac{1}{n})^2 - (x + \frac{1}{m})^2\right| \\ &= \left|(\frac{2}{n} - \frac{2}{m})x + \frac{1}{n^2} - \frac{1}{m^2}\right| \\ &= \left|(\frac{2}{N} - \frac{2}{2N})N + \frac{1}{N^2} - \frac{1}{4N^2}\right| \\ &= 1 + \frac{3}{4N^2} > 1 = \epsilon_0\end{aligned}$$

   Therefore, f_n^2 does not converge uniformly on $\mathbb{R}$.

4. (P.247 Q23) Since f_n, g_n converges uniformly to f, g respectively on A, and that f_n, g_n are bounded for all $n \in \mathbb{N}$, there exists $B, C \in \mathbb{R}$ such that $\|f\|_A \leq B$ and $\|g\|_A \leq C$ (Why?). To show $f_n g_n$ converges uniformly to fg on A, we use the definition of uniform convergence:
   Let $0 < \epsilon < 1$ be given, by Lemma 8.1.8, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|f_n - f\|_A < \frac{\epsilon}{2(1+C)}$ and $\|g_n - g\|_A < \frac{\epsilon}{2B+1}$. In particular, $\|g_n\|_A \leq \epsilon + C < 1 + C$

   Then for all $x \in A$, $n \geq N$,

   $$\begin{aligned}|f_n g_n(x) - fg(x)| &\leq |f(x)||g(x) - g_n(x)| + |g_n(x)||f(x) - f_n(x)| \\ &< B \cdot \frac{\epsilon}{2B+1} + (1+C) \cdot (\frac{\epsilon}{2(1+C)}) \\ &< \epsilon\end{aligned}$$

   Therefore, $f_n g_n$ converges uniformly to fg on A.

   Remark: Many students use the boundness of each function of the sequence (f_n) (similarly for (g_n)) to argue that there exists $M \in \mathbb{R}$ (independent of n) such that $\|f_n\|_A \leq M$ for all $n \in \mathbb{N}$. This is not true in general (consider $f_n(x) \equiv n$ on $\mathbb{R}$) unless (f_n) converges uniformly to some function on A. One has to use Cauchy criterion to argue the existence of such M.